Randomized Space-Time Coding for Distributed Cooperative Communication

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Motivation

Problem: Decentralizing the Space-time (ST) coding policy:

- Orthogonal space-time codes are proposed to achieve diversity by Laneman et al [2003] ⇒ codes have to be assigned to each node.

- **Proposed scheme**: Randomized space-time codes ⇒ each node uses a random linear combination of the columns of an underlying ST code.
Let $s = [s_0 s_1 \ldots s_{n-1}]$ be the block of source symbols

$\mathcal{G}(s): P \times L (P \geq L)$ underlying space-time code ($L$ virtual antennas)

**IDEA:** The randomized space-time coding is a double mapping:

$$s \rightarrow \mathcal{G}(s) \rightarrow \mathcal{G}(s)\mathcal{R},$$

$\mathcal{R} = [r_1 \ldots r_N]$, $r_i$ is the random coefficient vector for the $i$’th node.

$r_i$’s are independent $\Rightarrow$ decentralized

The received signal at the destination is

$$y = \mathcal{G}(s)\mathcal{R}h + w,$$

$w$ is the AWGN and $h$ is the channel vector.
Example: randomized Alamouti

ST code is Alamouti: \( P = 2, L = 2, \ G = \begin{bmatrix} s_1 & s_2 \\ s_2^* & -s_1^* \end{bmatrix} \) and \( N = 2 \).

- Uniform phase randomization:
  \[ \mathcal{R}_1 = \begin{bmatrix} e^{j\theta_{11}} & e^{j\theta_{12}} \\ e^{j\theta_{21}} & e^{j\theta_{22}} \end{bmatrix} \]
  where \( \theta_{kn} \sim i.i.d. U(-\pi, \pi) \), \( n, k \in \{1, 2\} \).

- Selection randomization:
  \[ \mathcal{R}_2 = [r_1 \ r_2], \ r_i \in \{[1 \ 0]^t, [0 \ 1]^t\} \]
  with equal probability.

Statistics of \( \mathcal{R} \leftrightarrow \) performance
**Definition:** The diversity order $d^*$ of a scheme with probability of error $P_e(SNR)$ is defined as

$$d^* = \lim_{SNR \to \infty} \frac{-\log P_e(SNR)}{\log SNR}.$$  

(1)

Given $R_{L \times N}$, the randomization matrix $\Rightarrow d^* \leq r \triangleq \min\{L, N\}$

**AIM:** To find sufficient conditions such that diversity $r$ is achieved.

Let $M = \{s_1, s_2, \ldots, s_{|M|}\} \triangleq$ message set, and $g_i \triangleq g(s_i)$.

Let $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_N$ be $\text{eig}(A)$, $A = A^H$, $|A|_{k+} = \prod_{\ell=N-k+1}^{N} \lambda_\ell$. 


Theorem 1 \hspace{1em} \textit{Assume the scheme satisfies the conditions:}

C1) \textit{Rank Criterion for } \mathcal{G} : \forall \{\mathcal{G}_k, \mathcal{G}_i\}, (\mathcal{G}_k - \mathcal{G}_i) \text{ is of rank } L.

C2) \textit{Rank Criterion for } \mathcal{R} : \mathcal{R} \text{ is full-rank with probability } 1.

\textit{Then, under coherent decoding } P_e \textit{ is bounded as}

\[
P_e \leq \frac{4^{-r}(|\mathcal{M}| - 1) \text{SNR}^{-r}}{\min_{(i,j)} \left\{ |(\mathcal{G}_i - \mathcal{G}_j)^H (\mathcal{G}_i - \mathcal{G}_j)|_r \right\}} \mathbb{E} \left\{ \frac{1}{|\mathcal{R} \mathcal{R}^H|_r} \right\}. \tag{2}
\]

- C1 is equivalent to the rank criterion for deterministic space-time code \cite{Tarokh1998} \Rightarrow choose \mathcal{G} designed for a multi-antenna system.

- C2 is satisfied if the columns of \mathcal{R} are independently drawn from a continuous distribution.
Finiteness of \( \mathbb{E} \left\{ \frac{1}{|\mathbf{R}\mathbf{R}^H|} \right\} \)

**Theorem 2** Let \( \mathbf{R} \) be an \( L \times N \) random matrix, and \( r = \min\{L, N\} \).

1. Let \( p(\mathbf{R}) \) be the bounded probability density function of \( \mathbf{R} \).

2. Assume that \( \text{Tr}(\mathbf{R}\mathbf{R}^H) \leq P_T \) with probability 1 (transmission power constraint).

\[
|N - L| \geq 1 \Rightarrow \mathbb{E}\{|\mathbf{R}\mathbf{R}^H|_{r+}^{-1}\} < \infty.
\]

**Definition:** A non-negative function \( f(\text{SNR}) = \Theta(x^\alpha) \) as \( x \to 0 \) if \( \exists \epsilon > 0 \) and \( 0 < c_1 < c_2 \) such that \( |x| < \epsilon \) implies \( c_1 x^\alpha \leq f(\text{SNR}) \leq c_2 x^\alpha \)

**Lemma 1** \( F(x) = \Pr\{|\mathbf{R}\mathbf{R}^H|_{r+} \leq x\} : F(x) = \Theta(x^\alpha) \) around \( x = 0 \) for some \( \alpha \in R \). Then,

\[
\mathbb{E}\{|(\mathbf{R}\mathbf{R}^H)_{r+}^{-1}\} < \infty \iff \alpha > 1.
\]
Diversity order of randomized ST

MAIN OBSERVATION (Thm. 2): If the rank conditions C1 and C2,
\( p(\mathcal{R}) < \infty \) and \( \text{Tr}(\mathcal{R}\mathcal{R}^H) < \infty \) with probability 1, then

\[
\begin{align*}
d^* &= \begin{cases} 
N & \text{if } L \geq N + 1 \\
L & \text{if } L \leq N - 1 
\end{cases} 
\end{align*}
\] (4)

For \( N = L \), we naturally expect that \( N - 1 \leq d^* \leq N \).

Remarks:

- If we choose \( L \) large enough, randomized space-time codes achieve diversity order equal to the number of nodes \( N \).
- It is sufficient to have 1 more node than \( L \) to have full diversity \( L \).
Let $r_k$ be the $k$’th column of the randomization matrix $\mathcal{R}$.

Gaussian Randomization: The elements of the randomization matrix $\mathcal{R}$ are zero-mean independent and complex Gaussian.

Uniform Phase Randomization: $r_k = a_k[e^{j\theta_i[0]}, \ldots, e^{j\theta_i[L]}]^t$ where each $\theta_i[N] \overset{iid}{\sim} U(0, 2\pi)$ and $a_k \overset{iid}{\sim} U(1 - \epsilon, 1 + \epsilon)$ for some small $\epsilon > 0$, $\theta_i[N]$’s are independent of $a_k$’s.

Uniform distribution on a complex hypersphere: $r_k$ is uniformly selected on the surface of a complex hypersphere of radius $\rho$, i.e., $\|r_k\| = \rho$. 
Simulations

Alamouti scheme with $N = 3$ and $N = 10$
Conclusions

- Randomized ST coding can truly decentralize the use of space-time coding in distributed networks.

- Different designs can provide the diversity order ($\min(N, L)$) when number of nodes $N$ is different than the number of virtual antennas $L$. For $N = L$, diversity order can be fractional.

- The randomized schemes achieve the performance of a centralized space-time code in terms of coding gain as the number of nodes increases.